FLOW OF A CONDUCTING FULID PAST A ROTATING CONDUCTING CYLINDER

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ABSTRACT The problem of finding the drag and torque on a cylinder immersed in an inviscid liquid and steadily rotating about its axis (z-axis) is discussed when both the liquid and the cylinder are conducting and the magnetic field and the streaming motion at infinity are uniform and parallel to the x-axis The magnetic Reynolds number R_M is assumed small and the first order effect of the conductivity (R_M) is studied Explicit expressions for the coefficients of drag and torque and the vorticity are obtained It is found that the Maxwell stresses do not contribute to the drag coefficient, but make non-zero contribution to the torque. The variation of the vorticity on circles concentric to the circular section of the cylinder has been shown graphically when the radii of the circles are 1 05, 1 20, 1 40, 1.60 times the radius of the cylinder

INTRODUCTION

The problem of estimating the effect of a uniform external magnetic field on a flow past a sphere or a body of revolution which at infinity, along with the magnetic field, is parallel to the axis of symmetry, has been discussed by Chester (1957, 1961) under various assumptions. Ludford and Murray (1960) have discussed the flow of an inviscid and finitely conducting liquid past a magnetized sphere for small values of the dimensionless parameter β representing the ratio of some standard magnetic pressure to the free stream dynamic pressure. Recently, Murray and Chi (1960) have considered the corresponding problem for the case of a magnetized cylinder placed in a uniformly streaming fluid whose direction at infinity is normal to the axis of the cylinder

In the present note we have studied the flow characteristic of a conducting fluid past a conducting rotating cylinder under the assumption that the flow and magnetic field at infinity are uniform and normal to the axis of the cylinder. In particular, we have found expressions for the drag and toraue coefficients of the cylinder and the distribution of vorticity in the case when the magnetic Reynolds number is small. We find that (i) the Maxwell stresses do not contribute to the drag though the perturbed pressure makes a non-zero contribution, (ii) the Maxwell stresses produce a toraue proportional to the angular velocity Ω and tend to oppose the motion, (iii) the vorticity vanishes on the axis of symmetry drawn upstream and (iv) the conductivity of the rotating cylinder affects the flow characteristics.

The analysis is carried out allowing for the difference in the conductivities of the liquid and the cylinder, though the magnetic Reynolds number for the liquid and the cylinder have been assumed to be of the same order of magnitudes.

BASIC EQUATIONS

We consider the steady two dimensional flow of an incompressible inviscid and electrically conducting liquid. The basic equations in the usual notation are*

$$(\text{curl } \underline{q})\times\underline{q} = -\text{grad}\,(p/\rho+\tfrac{1}{2}q^2)+\frac{\mu}{\rho}(\text{curl } \underline{H})\times\underline{H}, \qquad \ldots (1)$$

$$\text{div } \underline{q} = 0, \qquad \ldots (2)$$

$$\underline{J} = \text{curl } \underline{H} = \sigma[\underline{E}+\mu\underline{q}\times\underline{H}], \qquad \ldots (3)$$

$$\text{curl } \underline{E} = 0, \qquad \text{div } \underline{E} = 0 \qquad \ldots (4, 5)$$

$$\text{div } \underline{H} = 0. \qquad \ldots (6)$$

We shall take the axis of the cylinder as z-axis and the undisturbed direction of flow and the external magnetic field as the x-axis. We can neglect (5) from our discussion as it is identically satisfied in view of (3) and the fact that $\underline{q}\times\underline{H}$ is along the z-axis. From (4) and (5) we find that $\underline{E}$ vanishes identically.

Using the radius a of the cylinder, the uniform velocity U at infinity and the magnitude h of the magnetic field at infinity as the standard quantities, the equations (1)-(6) reduce to the following dimensionless forms (retaining the same symbols for the dimensionless puantities as for the nondimensionless ones.)

$$(\text{curl } \underline{q})\times\underline{q} = -\text{grad } P+\beta(\text{curl } \underline{H})\times\underline{H}, \qquad \ldots (7)$$

$$\text{div } \underline{q} = 0 \qquad \ldots (8)$$

$$\text{curl } \underline{H} = R_M\underline{q}\times\underline{H}, \quad \text{div } \underline{H} = 0, \qquad \ldots (9, 10)$$

where

$$\beta = \frac{\mu h^2}{\rho U^2}, \quad P = p+\tfrac{1}{2}q^2, \quad R_M = \mu\sigma a U.$$

We shall assume that the magnetic permeabilities of the cylinder and the liquid are the same and equal to μ. When the liquid is non-conducting i.e. $R_M = 0$, the flow outside the cylinder is the familiar potential flow defined by

$$\underline{q}_0 = \left[\left(1-\frac{1}{r^2}\right)\cos\theta, \quad -\left(1+\frac{1}{r^2}\right)\sin\theta, \quad 0\right] \qquad \ldots (11)$$

in the cylinderical coordinates (r, θ, z).

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*Throughout the paper m.k.s system has been used.

When R_M is non-zero but small we expend p, q and $\underline{H}$ in powers of R_M :

$$p = p_0+R_M p_1+R^2_M p_2+\ldots \quad ,$$
$$\underline{q} = \underline{q}_0+R_M \underline{q}_1+R^2_M \underline{q}_2+\ldots \quad ,$$
$$\underline{H} = \underline{H}_0+R_M \underline{H}_1+R^2_M \underline{H}_2+\ldots , \qquad \ldots \quad (12)$$

where $\underline{H}_0$ is obviously the magnetic field at infinity.

SOLUTION OF THE FIRST ORDER PERTURBATION EQUATION

The first order perturbations p_1. q_1, $\underline{H}_1$ are determined by the following equations .

$$(\text{curl } \underline{q}_1)\times \underline{q}_0 = -\text{grad } P_1+\beta\,(\text{curl } \underline{H}_1)\times \underline{H}_0, \qquad \ldots \quad (13)$$

$$\text{div } \underline{q}_1 = 0, \qquad \ldots \quad (14)$$

$$\text{curl } \underline{H}_1 = q_0 \underline{H}_0 \;; \quad \text{div } \underline{H}_1 = 0. \qquad \ldots \quad (15,\ 16)$$

The equations (15) and (16) also hold for inside the cylinder if we take $q_0 = (0, \Omega\, r, 0)$.

In view of (16) we take

$$\underline{H}_1 = \left(\frac{1}{r} \frac{\partial A}{\partial \theta}, -\frac{\partial A}{\partial r}, 0 \right). \qquad \ldots \quad (17)$$

and then (15) reduces to

$$\nabla^2 A = -H_0 \sin 2\theta \qquad \ldots \quad (18)$$

Since we have taken h as the standard magnetic field, $|\,H_0\,| = 1$ in dimensionless form. The solution of (18) which vanishes at infinity is given by

$$A = \sum_{m=1}^{\infty} r^{-m}\,(C_m \cos m\theta + D_m \sin m\theta)+\frac{H_0 \sin 2\theta}{4}. \qquad \ldots \quad (19)$$

Defining the magnetic vector potential inside the cylinder by $(0, 0\, \bar{A})$, we have

$$\bar{A} = \sum_{m=1}^{\infty} r^{m}(A_m \cos m\theta + B_m \sin m\theta)+\frac{\Omega' H_0 r^3 \cos \theta}{8}, \qquad \ldots \quad (20)$$

where

$$\Omega' = \sigma'\Omega$$

σ' being the conductivity of the cylinder. Continuity of the normal and tangential components of the magnetic fields on the surface ($r = 1$) of the cylinder gives

$$\left.\begin{aligned} A_m = B_m = C_m = D_m = 0 \quad (m \neq 1, 2) \\ B_1 = D_1 = A_2 = C_2 = 0, \end{aligned}\right\} \qquad \dots \quad (21)$$

and

$$\begin{aligned} A_1 &= -\frac{\Omega' H_0}{4}, \quad C_1 = -\frac{\Omega' H_0}{8} \\ B_2 &= \frac{H_0}{8} \quad D_2 = -\frac{H_0}{8} \end{aligned} \qquad (22)$$

Thus

$$A = -\frac{\Omega' H_0 \cos\theta}{8r} + \frac{H_0 \sin 2\theta}{4}\left(1 - \frac{r^{-2}}{2}\right), \qquad \dots \quad (23)$$

and

$$\bar{A} = -\frac{\Omega' H_0}{4}\left(r - \frac{r^3}{2}\right)\cos\theta + \frac{H_0 r^2 \sin^2\theta}{8}, \qquad \dots \quad (24)$$

The equations (23) and (24) determine the magnetic field completely.

CHARACTERIZATION OF THE FLOW

Taking the curl of (13) and using (14) we have

$$\left(1 - \frac{1}{r^2}\right)\cos\theta\frac{\partial\xi}{\partial r} - \frac{1}{r}\left(1 + \frac{1}{r^2}\right)\sin\theta\frac{\partial\xi}{\partial\theta} = -\frac{2H_0^2\beta\sin 3\theta}{r^3} \qquad \dots \quad (25)$$

where

$$\text{curl } \underline{q_1} = \xi e_z, \qquad \dots \quad (26)$$

is the vorticity vector.

For the potential flow ($R_M = 0$), the velocity potential ϕ_0 and the stream function ψ_0 are given by

$$\phi_0 = \left(r + \frac{1}{r}\right)\cos\theta, \quad \psi_0 = \left(r - \frac{1}{r}\right)\sin\theta. \qquad \dots \quad (27)$$

In view of (27), (25) reduces to

$$q_0^2\frac{\partial\xi}{\partial\phi_0} = -\frac{2H_0^2\beta\sin 3\theta}{r^3}, \qquad \dots \quad (28)$$

so that

$$\xi = -2H_0^2\beta\int_{-\infty}^{\phi_0(P)}\frac{\sin 3\theta}{q_0^2 r^2}\, d\phi_0, \qquad \dots \quad (29)$$

as the vorticity vanishes upstream as $r \to \infty$ and the integration is carried out along a stream line ψ_0 = constant and $\phi_0(P)$ is the velocity potential at the point P.

Along a stream line ψ_0 = constant, we have from (27)

$$\left(1+\frac{1}{r^2}\right)\sin\theta\, dr + \left(r-\frac{1}{r}\right)\cos\theta\, d\theta = 0.$$

Also

$$d\phi_0 = \left(1-\frac{1}{r^2}\right)\cos\theta\, dr - \left(r+\frac{1}{r}\right)\sin\theta\, d\theta$$

$$\frac{r q_0^2 d\theta}{\left(1+\frac{1}{r^2}\right)\sin\theta} \qquad (30)$$

along a streamline on using the last equation, hence from (29) we have

$$\xi = 2H_0^2\beta \int_{\phi=-\infty}^{\phi_0(P)} \frac{\sin 3\alpha}{(1+r^2)\sin\alpha}\, d\alpha \qquad \ldots \quad (31)$$

From (27) we obtain

$$\frac{\psi_0 \pm \sqrt{\psi_0^2+4\sin^2\theta}}{2\sin\theta} \qquad \ldots \quad (32)$$

If $0<\theta<\pi$, ψ_0 is positive and we take the positive sign in (32), When $-\pi<\theta<0$, the negative sign has to be taken in (32). Hence

$$\xi = 8H_0^2\,\beta \int_{\theta=\pi}^{\theta} \frac{\sin\alpha\,\sin 3\alpha\, d\alpha}{[4\sin^2\alpha+\{\psi_0+\sqrt{\psi_0^2+4\sin^2\alpha}\}^2]} \qquad \ldots \quad (33)$$

for a point on a streamline for which $y>0$ and similarly for a point on a streamline for which $y<0$

$$\xi = 8H_0^2\beta \int_{\theta=-\pi}^{\theta} \frac{\sin\alpha\,\sin 3\alpha\, d\alpha}{[4\sin^2\alpha+\{\psi_0-\sqrt{\psi_0^2+4\sin^2\alpha}\}^2]}. \qquad \ldots \quad (34)$$

The boundedness of vorticity can easily be established from (33) and (34). From (33) we have

$$|\xi| \leqslant \frac{2H_0^2\beta(\pi-\theta)}{\psi_0^2} \qquad \ldots \quad (35)$$

and from (34)

$$|\xi| \leqslant \frac{2H_0^2\beta(\pi+\theta)}{\psi_0^2} \qquad \ldots \quad (35)$$

Putting $\psi_0 = 0$ in (33) and (34) and performing the integration we have

$$\left.\begin{aligned} |\xi| &\leqslant 6H_0^2\beta(\pi-\theta), \quad \text{when } y>0, \\ \text{and} \quad |\xi| &\leqslant 6H_0^2\beta(\pi+\theta), \quad \text{when } y<0. \end{aligned}\right\} \quad \ldots \quad (36)$$

When ψ_0 is large,

$$\xi \simeq \frac{2H_0^2\beta}{\psi_0^2}\left[\frac{\sin 2\theta}{4} - \frac{\sin 4\theta}{8}\right]$$

Equations (33) and (34) show that at two points (r, θ) and $(r, -\theta)$, vorticity has same magnitude but opposite sign.

In terms of the elliptic integrals of the first and second kind, from (33) we have

$$\xi = H_0^2\beta.\left[(\theta-\pi+\sin 2\theta)-\psi_0\left\{\frac{2\psi_0}{k'}E(\pi/2, k)-\frac{\psi_0}{k'}\left(E(\psi, k)-\frac{k^2\sin\psi\cos\psi}{\sqrt{1-k^2\sin^2\psi}}\right)\right.\right.$$

$$\left.\left.+(3+\psi_0^2)\left(-\frac{2k'F(\pi/2, k)}{\psi_0}+\frac{1}{\psi_0}[k'F(\psi, k)]\right.\right. \quad (37)$$

for $\quad 0 \leqslant \theta \leqslant \pi/2 \quad$ and

$$\xi = H_0^2\beta\left[(\theta-\pi+\sin 2\theta)-\psi_0\left\{\frac{\psi_0}{k'}E(\gamma, k) - \frac{\psi_0 k^2}{k'}\frac{\sin\psi\cos\psi}{\sqrt{1-k^2\sin^2\psi}}\right.\right.$$

$$\left.\left.-(3+\psi_0^2)\frac{k'}{\psi_0}F(\psi, k)\right\}\right] \quad (38)$$

when $\quad \pi/2 \leqslant \theta < \pi$, and where

$$n = \frac{2}{\psi_0}, \quad \sin\psi = \frac{\sqrt{1+n^2}}{\sqrt{1+n^2\sin^2\theta}}\sin\theta, \quad k^2 = \frac{4}{\psi_0^2+4}$$

and

$$k' = \sqrt{1-k^2}. \quad (39)$$

We plot in Fig. 1 the variation of ξ with θ for various values of r. We find that on a circle concentric with the cross-section of the cylinder the vorticity shows a considerable variation.

Since ξ is an odd function of θ we assume

$$\xi = \sum_{n=1}^{\infty} \xi_n(r)\sin n\theta. \quad \ldots \quad (40)$$

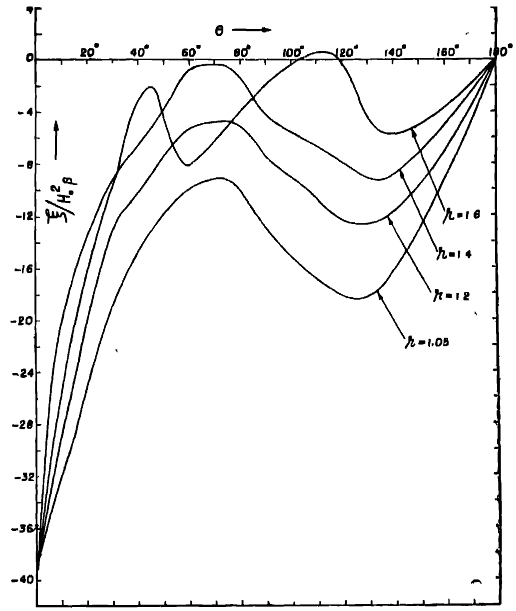

Fig. 1.

Substituting for ξ from (40) in (25) we get

$$\left(r-\frac{1}{r}\right)\frac{d\xi_{n+2}}{dr}+(n+2)\left(1+\frac{1}{r^2}\right)\xi_{n+2}$$

$$=-\left(r-\frac{1}{r}\right)\frac{d\xi_n}{dr}+n\left(1+\frac{1}{r^2}\right)\xi_n, \tag{41}$$

when $n \neq 2$, and when $n=2$, we have

$$\left(r-\frac{1}{r}\right)\frac{d\xi_4}{dr}+4\left(1+\frac{1}{r^2}\right)\xi_4=-\frac{4H_0{}^2\beta}{r^2}, \quad .. \quad (42)$$

as we can show that $\xi_2=0$.

On integrating (41) we have

$$\left(r-\frac{1}{r}\right)^{n+2}\xi_{n+2}=\int_1^r\left[-\left(r-\frac{1}{r}\right)^{n+2}\frac{d\xi_n}{dr}+n\left(r-\frac{1}{r}\right)^{n+1}\left\{1+\frac{1}{r^2}\right\}\xi_n\right]dr \quad .. \quad (43)$$

Integrating (42) we get ξ_4 explicitly in terms of r :

$$\xi_4=-\frac{H_0{}^2\beta(2r^6+3r^4-6r^2+1-12\,(\log r)r^4)}{(r^2-1)^4}, \quad ... \quad (44)$$

We can evaluate $\xi_6, \xi_8, \ldots$ similarly.

ξ_1 cannot be fixed up from this consideration, the recurrence relation (43) is of no use to evaluate $\xi_3, \xi_5, \ldots \xi_{2n+1}, \ldots$ However, the odd Fourier coefficients of ξ in (40) can be computed numerically from the expression for ξ, given by equations (37) and (38). Table 1 gives the values of $\xi_1/8H_0^2\beta$ for $r = 1\ 05$, 1.20, 1.40 and 1 60.

TABLE 1

$\dfrac{\xi_1}{8H_0^2\beta}$	$-0\ 544$	$-0\ 242$

The perturbed velocity $\underline{q}_1$ is determined by the stream function ψ

$$\underline{q}_1 = \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta},\ -\frac{\partial\psi}{\partial r},\ 0\right). \qquad \ldots \quad (45)$$

where ψ is determined by

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = -\xi. \qquad \ldots \quad (46)$$

Assuming $\qquad \psi = \sum_{n=1} \psi_n(r)\sin n\,\theta$

we have

$$\frac{d^2\psi_n}{dr^2}+\frac{1}{r}\frac{d\psi_n}{dr}-\frac{n^2}{r^2}\psi_n = -\xi_n. \qquad \ldots \quad (47)$$

In evaluating the drag coefficient we shall require only $\psi_2(r)$. We note that $\xi_2 = 0$ and $\dfrac{\partial\psi}{\partial r}, \dfrac{\partial\psi}{\partial\theta}$ have to be regular at infinity. The only admissible solution of (47) when $n = 2$ is $\psi_2 = 0$.
When $n = 4$,

$$\psi_4 = Ar^{-4}+H_0^2\beta\left[-\frac{1}{8}-\frac{5}{4}r^{-2}-\frac{5}{4}r^{-6}+\ldots\right.$$
$$\left.+(\log r)(r^{-2}-\frac{5}{2}r^{-4}-6r^{-6}+\ldots)+3r^{-4}(\log r)^2\right] \qquad \ldots \quad (48)$$

Since the normal component of velocity on the cylinder should vanish, we get $\psi_4 = 0$ at $r = 1$. This condition determines A. Similarly $\psi_6, \psi_8 \ldots$ are determined.

DRAG AND TORQUE ON THE CYLINDER

The force on the cylinder is composed of two parts, one due to the fluid pressure and the other due to the Maxwell stresses The θ-component of (13) gives, on integrating w.r.t.θ,

$$P_1 = -\frac{\beta H_0^2 \cos 3\theta}{6} - \frac{\beta H_0^2 \cos\theta}{2} + a_0 \tag{49}$$

at $r = 1$ where a_0 is a constant.

But

$$P_1 = p_1 + \underline{q}_0 \cdot \underline{q}_1 \tag{50}$$

and $\underline{q}_0$ on the cylinder is $(0, -2\sin\theta, 0)$.

We have

$$q_{1\theta} = -\sum \frac{d\psi_n}{dr} \sin n\theta \tag{51}$$

for the θ-component of q_1. The drag coefficient, D_p, due to the pressure is given by

$$D_p = -R \int_{\theta=0}^{2\pi} p_1 \cos\theta \, d\theta$$

$$= \frac{\pi H_0^2 R_M \beta}{2} \quad \ldots \quad (52)$$

The drag and torque on the cylinder due to the Maxwell stresses $T_{ij} = \mu H_i H_j - \frac{1}{2}\mu H^2 \delta_{ij}$. which provides a force on the cylinder with components.

$$\mu\left\{\tfrac{1}{2}(H_r^2 - H_\theta^2),\ H_r H_\theta,\ 0\right\}$$

The drag on the cylinder is given in terms of the non-dimensional parameter

$$D = \mu R \int_0^{2\pi} \{(H_{r0}H_{r1} - H_{\theta 0}H_{\theta 1})\cos\theta - (H_{r0}H_{\theta 1} + H_{r1}H_{\theta 0})\sin\theta\} d\theta = 0. \quad \ldots \quad (53)$$

In (53) H_{r0}, $H_{\theta 0}$ are the r- and θ-components of $\underline{H}_0$ and $R_M H_{r1}$ and $R_M H_{\theta 1}$ are the corresponding components of the perturbed magnetic field.

The Maxwell torque is given in terms of the dimensionless parameter T_m where

$$T_m = \mu R_M \int_0^{2\pi} (H_{r0} H_{\theta 1} + H_{r1} H_{\theta 0}) d\theta$$

$$= -\frac{\pi\mu\Omega' H_0^2 R_M}{4}. \quad \ldots \quad (54)$$

The negative sign shows that the torque tends to oppose the motion of the cylinder.

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